Why optimal diversification cannot outperform naive diversification: Evidence from tail risk exposure

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Abstract

This paper examines the outperformance of naive diversification relative to optimal diversification. From out-of-sample analysis using portfolios consisting of individual stocks as well as diversified equity portfolios, we find that optimal diversification fails to *consistently* outperform naive diversification. Our results show that naive diversification increases tail risk measured by skewness and kurtosis and makes portfolio returns more concave relative to equity benchmarks. In addition, tail risk exposure and concavity increases with the number of stocks in the portfolio. These results imply that the outperformance of naive diversification relative to optimal diversification represents a compensation for the increase in tail risk and the reduced upside potential associated with the concave payoff.

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1. Introduction

Although Markowitz's (1952) mean–variance framework provides the basic concept of modern portfolio theory and is still widely used in practice today in asset allocation and active portfolio management,[1] individual investors tend to use naive diversification rather than sophisticated diversification. For example, Benartzi and Thaler (2001) and Liang and Weisbenner (2002) find that investors follow the naive 1/*N* strategy to allocate their wealth across assets. Huberman and Jiang (2006) document that participants tend to allocate their contributions evenly across the funds they use, with the tendency weakening with the number of funds used.

The naive 1/*N* diversification rule is the strategy in which a fraction 1/*N* of wealth is allocated to each of the *N* assets available for investment at each rebalancing date. The naive strategy does not align with the mean–variance framework of optimal asset allocation strategy, which suggests giving more weight to those assets that contribute to higher mean–variance efficiency. Compared with the optimal portfolio, the most appealing feature of the 1/*N* portfolio is that it is easy to implement because it does not require any estimation of the moments of asset returns, optimization, and short sales. Furthermore, previous literature documents that optimal portfolio strategy does not dominate the naive 1/*N* strategy. For instance, Bloomfield, Leftwich, and Long (1977) show that sample-based mean–variance optimal strategies do not outperform a simpler strategy of maintaining equal dollar investments in each available stock. Jorion (1991) finds that the equally weighted and value-weighted indices have out-of-sample performance

[1] See Grinold and Kahn (1999), Litterman (2003), and Meucci (2005) for practical applications of the mean–variance framework. For a general survey of the literature on portfolio selection, see Campbell and Viceira (2002) and Brandt (2010).

similar to that of the minimum-variance and tangency portfolios obtained with Bayesian shrinkage methods.

In the literature on optimal portfolio choice, the outperformance of the 1/*N* portfolio strategy relative to the optimal portfolio strategy in out-of-sample asset allocation tests is largely attributed to estimation error in the optimal portfolio strategy. To implement the optimization model in practice, model parameters, such as the vector of expected excess returns over the risk-free rate and the variance–covariance matrix of asset returns, have to be estimated from the data. However, due to estimation error,[2] the estimated optimal portfolio rule can substantially differ from the true optimal rule. In other words, the estimation error in the optimal portfolio strategy produces extreme weights that fluctuate substantially over time and results in poor out-of-sample performance. For this reason, academic research proposes various extensions of the Markowitz model to reduce estimation errors with the goal of improving the performance of the Markowitz model.[3]

However, despite the considerable effort required to handle estimation error in the optimal portfolio strategy, this approach does not consistently dominate naive diversification. Recently, DeMiguel, Garlappi, and Uppal (2009b) report that none of the sample-based mean–variance models and almost none of the sophisticated extensions of the Markowitz rule consistently outperform the 1/*N* strategy.[4]

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[2] Merton (1980) documents that a very long history of returns is required to obtain an accurate estimate of expected r eturns. In addition, Green and Hollifield (1992) and Jagannathan and Ma (2003) report that the estimate of the varia nce–covariance matrix is poorly behaved.

[3] For examples of the asset allocation models proposed to reduce estimation error, see Bawa, Brown, and Klein (197 9), Jorion (1986), Best and Grauer (1992), MacKinlay and Pastor (2000), Pastor (2000), Jagannathan and Ma (2003), Garlappi, Uppal, and Wang (2007), and Kan and Zhou (2007).

[4] This finding triggered a new wave of research that seeks to develop portfolio strategies superior to 1/*N* and to reaffi rm the practical value of portfolio theory (e.g., DeMiguel et al., 2009a; Tu and Zhou, 2011; Behr, Guettler, and True

Our objective in this paper is to explore why optimal diversification cannot outperform naive diversification. In contrast to the previous literature, we focus on the tail risk exposure of the 1/*N* strategy rather than the estimation error of the optimal strategy.[5] Goetzmann et al. (2007) argue that strategies can enhance their performance at the expense of increased tail risk without additional information by changing the distribution of future returns. For example, when portfolio payoffs are concave relative to a benchmark, portfolio returns can be enhanced merely by compensation for the (modest) increase in tail risk and the reduced upside potential associated with the concave payoff.[6] The 1/*N* strategy can lead to a concave pattern of returns relative to the equity benchmark because it is similar to a conservative long-term asset mix strategy that causes the investor to buy equities as the equity market falls and sell them when it rises.

In this paper, using individual stock data as well as diversified equity portfolio data, we compare the out-of-sample performances and tail risks of the naive and optimal portfolio strategies. We use the following performance and tail risk measures: the Sharpe ratio, certainty equivalent (CEQ) returns, turnover, manipulation-proof performance measures (MPPMs), skewness, kurtosis, value at risk (VaR), and expected shortfall (ES). We also compare the return distributions of the naive and optimal portfolio strategies. Last, we examine whether the naive strategy has a concave payoff relative to the benchmark.

Our first contribution is to show that optimal diversification fails to outperform naive diversification in any consistent way. Unlike the previous literature, we construct a portfolio by using individual stocks as well as diversified portfolios. Compared with the case of using

benbach, 2012; Kourtis, Dotsis, and Markellos, 2012). Since it is not the focus of this study, we do not discuss this is sue in detail.

[5] Recently, Pflug, Pichler, Wozabal (2012) explain that the relative success of the 1/*N* rule is the result of an inaccura te specification of the data generating process, that is, a lack of accuracy in modeling the distributions of the random asset returns, in a stochastic portfolio optimization context.

[6] For example, see Brown et al. (2006) and Brown, Gregoriou, and Pascalau (2012).

diversified portfolios, optimal diversification is more likely to outperform naive diversification in the case of using individual stocks because individual stocks have higher idiosyncratic volatility than diversified portfolios. Nevertheless, naive diversification outperforms optimal diversification in both cases. Second, our paper contributes to the literature on the effect of diversification on tail risk because we show that the 1/*N* diversification increases tail risk measured by skewness and kurtosis as the number of stocks in portfolio increases. Last, our paper contributes to the literature on optimal portfolio choice because we focus on the tail risk exposure of the 1/*N* strategy rather than the estimation error of the optimal strategy as a source of the outperformance of the 1/*N* strategy. Our results imply that the outperformance of naive diversification is due to compensation for the increase in tail risk and the reduced upside potential associated with the concave payoff.

The rest of the paper is organized as follows. Section 2 introduces two asset allocation models: naive diversification and optimal diversification. Section 3 describes the dataset and methodology, including portfolio construction and measures of performance and tail risk. Section 4 presents our main empirical results and Section 5 presents our conclusions.

2. Naive and optimal diversification

In this section, we introduce the asset allocation models employed in this paper. While we use the 1/*N* portfolio strategy as the naive diversification, we use the sample-based mean–variance portfolio strategy and the sample-based mean–variance portfolio strategy with short sale constraints as the optimal diversification.[7]

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[7] To confirm whether our main results depend on the asset allocation model considered, we also use the other model s considered by DeMiguel et al. (2009b). In Section 4.1 (the analysis using diversified equity portfolios), we report t

2.1 The portfolio choice problem

Consider the standard portfolio choice problem in which an investor chooses his optimal portfolio weight among N risky assets.[8] Let R_t be the N-vector of excess returns over the risk-free asset on the N risky assets available for investment at time t. We assume that R_t is independent and identically distributed over time and has a multivariate normal distribution with expected returns on the risky asset in excess of the risk-free μ_t and $N \times N$ variance–covariance matrix of returns $\sum_t$.

According to the standard mean–variance framework (Markowitz, 1952, 1959; Sharpe, 1970), the investor chooses his vector of portfolio weights invested in the N risky assets, x_t, to maximize the following quadratic expected utility function:

$$\max_{x_t} x_t^T \mu_t - \frac{\gamma}{2} x_t^T \Sigma_t x_t, \tag{1}$$

where the scalar γ is the investor's risk aversion parameter. The solution of the above problem is

$$x_t^* = \frac{1}{\gamma} \Sigma_t^{-1} \mu_t. \tag{2}$$

The vector of *relative* portfolio weights invested in the N risky assets at time t, w_t^*, is

$$w_t^* = \frac{x_t^*}{\left|1_N^T x_t^*\right|} = \frac{\Sigma_t^{-1} \mu_t}{\left|1_N^T \Sigma_t^{-1} \mu_t\right|}, \tag{3}$$

he results of all the models. In Section 4.2 (the analysis using individual stocks), however, we report only three mod els—the 1/N portfolio strategy, the sample-based mean–variance portfolio strategy, and the sample-based mean–vari ance portfolio strategy with short sale constraints—because the main empirical results are similar and qualitatively u nchanged.

[8] In the paper, we consider the portfolio with only risky assets instead of the overall portfolio, which consists of both risk-free and risky assets, because we want to focus on the effect of asset allocation alone. The overall portfolio wou ld contain the effect of market timing ability as well. In addition, since the optimization problem is expressed in term s of returns in excess of the risk-free rate, we do not need the constraint that the weights sum to one.

where 1_N is an N-vector of ones. The relative weight is normalized by the absolute value of the sum of the portfolio weights to preserve the direction of the portfolio position in the few cases where the sum of the weights on the risky assets is negative.

2.2 Naive 1/*N* portfolio

The naive 1/*N* strategy is a special estimator of w^* that can be expressed as

$$w_{1/N} = \frac{1}{N}. \tag{4}$$

This strategy ignores all data information and does not implement any optimization or estimation. From Equation (3), this strategy can also be considered a strategy that estimates the moments μ_t and $\sum_t$ with the restriction that μ_t is proportional to $\sum_t \mathbf{1}_N$ for all t, which implies that the expected returns are determined by total risk rather than systematic risk.

2.3 Sample-based mean–variance portfolio

To implement Markowitz's (1952) mean–variance model, the optimal portfolio weights are usually calculated by using a two-step procedure. First, the mean and covariance matrices of asset returns are estimated by their sample counterparts $\hat{\mu}$ and $\hat{\Sigma}$, respectively. Second, these sample estimates are simply plugged into Equation (3) to compute the optimal portfolio weights. Note that this portfolio strategy completely ignores the possibility of estimation risk.

2.4 Sample-based mean–variance portfolio with short sale constraints

The sample-based mean–variance portfolio with short sale constraints is obtained by solving Equation (1) with additional nonnegativity constraints on the portfolio weights.

DeMiguel et al. (2009b) note that imposing a short sale constraint on the sample-based mean–variance problem is equivalent to shrinking the expected return toward the average. Similarly, Jagannathan and Ma (2003) show that imposing a short sale constraint on the minimum-variance portfolio is equivalent to shrinking the extreme elements of the covariance matrix. This simple constraint deals with estimation error in the mean–variance portfolio very well.

3. Data and methodology

3.1 Data

This paper uses two primary datasets, diversified equity portfolios and individual stocks, to construct portfolios from the naive 1/*N* and optimal portfolio strategies. The first primary dataset, which we consider the diversified equity portfolios, is the set of portfolios from the Fama–French four-factor model (the “FF-4-factor” dataset of DeMiguel et al., 2009b). Specifically, this dataset consists of 20 size- and book-to-market portfolios and the MKT, SMB, HML, and UMD portfolios. The number of risky assets in this dataset is 24. Following Wang (2005) and DeMiguel et al. (2009b), of the 25 size- and book-to-market portfolios, we exclude the five portfolios containing the largest firms, because the MKT, SMB, and HML portfolios are almost a linear combination of the 25 Fama–French portfolios. This dataset is collected from Ken French’s website[9]. The second primary dataset, which we consider individual stocks, comprises the monthly individual stock returns from the Center for Research in Security Prices (CRSP). The CRSP database covers all stocks on the NYSE, Amex, NASDAQ, and NYSE Arca. Our

[9] http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

sample spans the period January 1963 to December 2011, for a total of 588 monthly observations.[10]

3.2 Portfolio construction

To compare the out-of-sample performance and tail risk of the naive and optimal portfolio strategies, we use a rolling-sample approach adopted from the work of DeMiguel et al. (2009b). Let T be the length of asset returns and M be the length of the estimation window for parameters in the asset allocation model (in our study, $M = 120$ months). In each month t, starting from $t = M + 1$, we estimate the vector of expected excess asset returns over the risk-free rate and the variance–covariance matrix of asset returns by using nonmissing return observations over the past M months. These estimated parameters are then used to calculate the relative portfolio weights in the portfolio of only risky assets. Then, we use these weights to obtain the portfolio return in month $t + 1$. This process is repeated by adding the return for the next period in the dataset and dropping the earliest return, until the end of the dataset is reached. For this reason, the outcome of this rolling-sample approach is a series of $T – M$ monthly out-of-sample returns generated by each of the portfolio strategies.

In the analysis using diversified equity portfolios, we use all 24 assets in the dataset to construct portfolios. However, to examine not only the comparison between naive and optimal diversification in terms of performance and tail risk but also the effect of diversification on performance and tail risk, we construct a portfolio consisting of randomly selected stocks in the analysis, using individual stocks across various numbers of stocks in the portfolio. The number of individual stocks in the portfolio, N, ranges from two to 50. For each N, N stocks are randomly

[10] Since our major finding may be an artifact of the financial crisis and excessive tail risk, our analysis uses a sample that does not include the period of the US subprime crisis. For further information, see Section 4.3.

selected B times to construct portfolios (in our study, B = 10,000). To obtain sensible measures of performance and tail risk for portfolios from the time-series regressions, we require that all N stocks that are randomly selected to construct a portfolio have at least 120 months of overlapping return history. Moreover, while we use all $T – M$ monthly portfolio returns in the analysis using diversified equity portfolios, we use only the first 120 months of portfolio returns in the analysis using individual stocks to avoid oversampling returns in the period of the US subprime crisis.

3.3 Measures of performance and tail risk

3.3.1 Performance measures

We compute the out-of-sample Sharpe ratio, CEQ returns, portfolio turnover, and MPPMs to measure portfolio performance. The *out-of-sample Sharpe ratio* of strategy k is defined as the sample mean of out-of-sample excess returns (over the risk-free asset), $\hat{\mu}_k$, divided by their sample standard deviation, $\hat{\sigma}_k$:

$$\widehat{\text{SR}}_k = \frac{\hat{\mu}_k}{\hat{\sigma}_k}. \tag{5}$$

The *CEQ return* is defined as the risk-free rate that an investor is willing to accept rather than take the risky portfolio from a particular strategy. The CEQ return of strategy k is calculated as

$$\widehat{\text{CEQ}}_k = \hat{\hat{\mu}}_k - \frac{\gamma}{2}\sigma_k^2, \tag{6}$$

where $\hat{\mu}_k$ and $\hat{\sigma}_k$ are the mean and variance, respectively, of out-of-sample excess returns for strategy k and γ is risk aversion. Our paper reports the results for the case of $\gamma = 1$.

Portfolio *turnover* is defined as the average sum of the absolute value of the trades across the *N* available stocks. The portfolio turnover of strategy *k* is calculated as

$$Turnover_k = \frac{1}{T-M}\sum_{t=1}^{T-M}\sum_{j=1}^{N}\left(\left|\hat{\hat{w}}_{k,j,t+1} - w_{k,j,t^+}\right|\right), \tag{7}$$

where $\hat{w}_{k,j,t+1}$ is the portfolio weight in stock *j* at time *t* under strategy *k*; $\hat{w}_{k,j,t^+}$ is the portfolio weight before rebalancing at time $t+1$ and $\hat{w}_{k,j,t+1}$ is the targeted portfolio weight after rebalancing at time $t+1$. The turnover defined above can be interpreted as the average percentage of wealth traded in each period.

Goetzmann et al. (2007) point out that single-valued performance measures such as the Sharpe ratio are subject to dynamic manipulation that enhances performance scores without additional information by changing the distribution of future returns. To avoid this, we use the MPPM proposed by Goetzmann et al. (2007) to measure portfolio performance. The MPPM of the portfolio for strategy *k*, $\hat{\Theta}_k$, is computed as

$$\hat{\Theta}_k = \frac{1}{(1-\rho)\Delta t}\ln\left(\frac{1}{T-M}\sum_{t=M+1}^{T}[(1+r_{k,t}+r_{f,t})/(1+r_{f,t})]^{1-\rho}\right), \tag{8}$$

where $r_{k,t}$ is the out-of-sample excess return for strategy *k* at time *t*, $r_{f,t}$ is the risk-fee rate at time *t*, Δt is the length of time between observations, and ρ is risk aversion. The MPPM in Equation (8) does not require any specific distribution of return. The estimated MPPM, $\hat{\Theta}$, is such that investors with a constant relative aversion ρ can earn $\exp(\hat{\Theta}\cdot\Delta t)$ above the risk-free rate from a risk-free portfolio. This measure penalizes negative excess returns more as relative risk aversion increases, since the scores also decrease, given observed return data, as ρ increases. In our paper, we report the results for the case of $\rho = 3$.

3.3.2 Tail risk measures

We compute the out-of-sample skewness, kurtosis, VaR, and ES to measure portfolio tail risk. The *skewness* and *kurtosis* of a portfolio for strategy *k* are computed as

$$\widehat{\mathrm{SK}}_k = \frac{E(r_k - \hat{\mu}_k)^3}{\hat{\sigma}_k^3}, \tag{9}$$

$$\widehat{\mathrm{KU}}_k = \frac{E(r_k - \hat{\mu}_k)^4}{\hat{\sigma}_k^4}, \tag{10}$$

where r_k is the out-of-sample excess return for strategy *k*, $\hat{\mu}_k$ is the mean of r_k, and $\hat{\sigma}_k$ is the variance of r_k. Skewness is a measure of the symmetry of the probability density function. A positive (negatively) skewed distribution has a larger and longer right (left) tail and more probability mass is concentrated on the left-hand (right-hand) side of the mean. Kurtosis is generally regarded as a measure of a distribution's tail heaviness relative to that of the normal distribution; however, it also measures a distribution's peakedness. Due to the higher-power terms in Equations (9) and (10), the measure of skewness and kurtosis tend to be influenced by one or more outliers in the data of interest.

The measures VaR and ES are widely used risk measures of the risk of loss on a specific portfolio of financial assets. The VaR is the potential maximum loss for a given confidence level $1-\alpha$ and the ES is the expected loss, conditional on the loss being greater than or equal to the corresponding VaR. We compute the VaR and ES of the portfolio for strategy *k* as

$$\Pr(r_k < -VaR_\alpha) = \alpha, \tag{11}$$

$$ES_\alpha = -E[r_k \mid r_k \le -VaR_\alpha], \tag{12}$$

where r_k is the out-of-sample excess return for strategy k and $1-\alpha$ is the confidence level. Our paper reports the results for the case of $\alpha = 0.05$. Compared with the VaR, the ES is more sensitive to the shape of the loss distribution in the tail of the distribution.

4. Empirical results

In this section, to investigate why optimal diversification cannot outperform naive diversification, we empirically compare naive diversification to optimal diversification in terms of performance, tail risk, return distribution, and portfolio return concavity. Section 4.1 reports the empirical results using diversified equity portfolios, the extended results of DeMiguel et al. (2009b),[11] whose sample period includes the recent US subprime crisis. Section 4.2 reports the empirical results using individual stocks.

4.1 Results from the analysis of diversified equity portfolios

4.1.1 Performance and tail risk measures

[Insert Table 1 about here]

Table 1 reports the performance and tail risk measures for a portfolio consisting of diversified equity portfolios across the asset allocation models considered by DeMiguel et al. (2009b). The asset allocation models are listed in Appendix 1.[12]

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[11] The sample period of DeMiguel et al. (2009b) is from July 1963 to November 2004.

[12] We construct portfolios by using a rolling-sample approach, except for the sample-based mean–variance in sample (mv - in sample).

Analyzing the performance measures, we find several results that are consistent with those of DeMiguel et al. (2009b). First, none of the strategies from the optimizing models consistently outperform the 1/*N* strategy. While the constrained strategies have higher Sharpe ratios than the 1/*N* strategy, none of the strategies from the optimal models is better than the 1/*N* strategy in terms of CEQ and turnover. In particular, the sample-based mean–variance strategy (mv) is worse than the 1/*N* strategy. In Table 1, while the Sharpe ratio, CEQ, and turnover of the 1/*N* strategy are 0.1393, 0.0056, and 0.0197, respectively, those of the sample-based mean–variance strategy (mv) are 0.0770, -0.4344, and 145.4396, respectively. Second, Bayesian strategies seem to improve performance, but not very effectively. While the Bayes–Stein strategy (bs) and the Bayesian data-and-model strategy (dm) outperform the sample-based mean–variance strategy (mv), the 1/*N* strategy outperforms these Bayesian strategies. Third, constrained policies help improve performance, but not sufficiently. Constrained strategies do better than their corresponding unconstrained strategies. For example, while the Sharpe ratio, CEQ, and turnover of the Bayes–Stein strategy (bs) are 0.0795, -0.1656, and 46.7942, respectively, those of the Bayes–Stein strategy with short sale constraints (bs-c) are 0.1571, 0.0057, and 0.2315, respectively. However, the constrained strategies do not consistently outperform the 1/*N* strategy. Although some of the constrained strategies, such as the minimum-variance strategy with short sale constraints (min-c) and minimum-variance with generalized constraints (g-min-c), have significantly higher Sharpe ratios than the 1/*N* strategy, none of the constrained strategies has a higher CEQ than the 1/*N* strategy in a statistically significant way. Furthermore, the 1/*N* strategy has the lowest turnover.

From analysis of the tail risk measures, first, we find that the strategy that exhibits better performance tends to have negative skewness and positive kurtosis. In particular, the skewness of the 1/*N* strategy and the constrained strategies, which outperform the other strategies, is less than -0.6 and the kurtosis of those strategies is greater than five. This result implies that the better – performing strategy tends to increase tail risk exposure. Second, the strategy that contains outliers from extreme portfolio weights tends to have extremely large skewness and kurtosis. For example, the skewness of the sample-based mean–variance strategy (mv) and the Bayes–Stein strategy (bs) is greater than 19 and their kurtosis is greater than 400. Third, the strategy that has a lower standard deviation tends to have lower VaR and ES. Specifically, the minimum-variance strategy (min) and the mixture of minimum-variance and 1/*N* strategies (ew-min) have the lowest VaR and ES values, as well as standard deviation.

4.1.2 Return distribution

To illustrate the difference in tail risk exposure across strategies, we compare the return distributions of the naive 1/*N* and optimal portfolio strategies.

[Insert Figure 1 about here]

Figure 1 reports the kernel smoothed histogram for the return distributions of portfolios consisting of both diversified stock portfolios from the naive 1/*N* strategy as well as optimal portfolio strategies. To simplify the graph, we consider only the sample-based mean–variance (mv) and the sample-based mean–variance with short sale constraints (mv-c) as the optimal portfolio strategy. To compare these return distributions to a normal distribution, in Figure 1 we also report the corresponding normal distribution generated by the pooled mean and pooled standard deviation from the return distributions of the naive 1/*N* and optimal portfolio strategies.

Figure 1 shows that all strategies exhibit leptokurtic distribution, which means positive excess kurtosis and heavier tails than the normal distribution. However, only the strategies that have better performance, the naive 1/*N* strategy and the sample-based mean–variance portfolio strategy with short sale constraints (mv-c), have negatively skewed distribution, which means a larger and longer left tail than the normal distribution. In other words, the strategy that has better performance tends to have increased left tail risk exposure and reduced upside potential. This tendency is clearer in the naive 1/*N* strategy than in the other strategies. This result implies that the strategy that has better performance, especially the naive 1/*N* strategy, may have a more concave payoff than the strategy with poor performance.

4.1.3 Concavity of the portfolio payoff

In this section, we examine whether the portfolio from the naive 1/*N* strategy or the optimal portfolio strategies has concave payoff relative to the equity benchmark. More specifically, to examine the concavity of a portfolio's payoff, we use the coefficients of Henriksson and Merton (1981) and Treynor and Mazuy (1966). Henriksson and Merton (1981) and Treynor and Mazuy (1966) propose a test of market timing ability in an extended market model regression since they argue that a fund manager who can successfully time the market should demonstrate convex returns relative to a benchmark. However, nonlinear return patterns are not limited to market timing strategies. For example, Agarwal and Naik (2004) argue that concave payoff patterns in hedge fund returns result from exposure to option-based risk factors rather than to negative timing ability. While intent to limit tail risk exposure through portfolio insurance–type strategies will lead to convex returns relative to a benchmark, increases in tail risk exposure will lead to concave returns.

The coefficients of Henriksson and Merton (1981) and Treynor and Mazuy (1966) are computed from the regression

$$r_t = \gamma_0 + \gamma_1 m_t + \gamma_2 c_t + \varepsilon_t, \tag{13}$$

where r_t and m_t are the excess returns on the portfolio and the market, respectively. The timing variable is $c_t = Max(-m_t, 0)$ for Henriksson and Merton (1981) and $c_t = m_t^2$ for Treynor and Mazuy (1966).

[Insert Table 2 about here]

Table 2 reports the timing coefficients for a portfolio consisting of diversified equity portfolios across the asset allocation models listed in Appendix 1. The t-statistics are given in parentheses. Numbers in bold denote the coefficients' statistical significance. In Table 2, we find that the strategies that show higher performance in Table 1—such as the naive 1/*N* strategy, MacKinlay and Pastor's (2000) missing-factor model (mp), the sample-based mean–variance with short sale constraints (mv-c), the Bayes–Stein strategy with short sale constraints (bs-c), and the minimum-variance strategy with generalized constraints (g-min-c)—have a concave payoff relative to the equity benchmark. In particular, these strategies have a significantly positive coefficient for the market portfolio (γ_1) and a significantly negative coefficient for the timing variable (γ_2). On the other hand, the strategies that show lower performance in Table 1—such as the sample-based mean–variance (mv) and the Bayes–Stein (bs)—seem to have a convex payoff rather than a concave payoff.

In summary, we find that none of the strategies from the optimizing models consistently outperforms the naive 1/*N* strategy in the analysis of diversified equity portfolios. A high-performance strategy tends to have increased left tail risk exposure, reduced upside potential, and a concave payoff relative to the equity benchmark. This tendency is clearer in the naive 1/*N* strategy than in the optimal portfolio strategies.

4.2 Results from the analysis of individual stock portfolios

4.2.1 Performance and tail risk measures

[Insert Figure 2 about here]

In this section, we use individual stocks to construct portfolios while we use diversified equity portfolios to construct the portfolios in Section 4.1. Figure 2 shows the mean value of the performance and tail risk measures for the naive 1/*N* portfolio, the sample-based mean–variance portfolio, and the sample-based mean–variance portfolio with short sale constraints.[13] To investigate not only the comparison between naive and optimal diversification in terms of performance and tail risk but also the effect of diversification on performance and tail risk, we plot the performance and tail risk measures as a function of the number of stocks in a portfolio. In addition to the mean value of measures, the dotted line in Figure 2 shows the confidence band of each measure. The range of the confidence band is between -2 and 2 standard deviations.

Analyzing the performance measures, we find several results that are consistent with the literature on optimal portfolio choice in the presence of estimation error. First, we find that the naive diversification strategy consistently outperforms the optimal diversification strategies. Regardless of the kind of performance measure or the number of stocks in a portfolio, the naive

[13] Appendix 2 reports the tables for the mean values of these measures.

1/*N* portfolio has a higher Sharpe ratio, CEQ return, MPPM, and lower turnover than the mean–variance portfolios. In particular, in the case of *N* = 30, the Sharpe ratio of the naive 1/*N* portfolio is 0.1921, while those for the sample-based mean–variance portfolio and the sample-based mean–variance portfolio with short sale constraints are -0.0079 and 0.0717, respectively. The CEQ return of the naive 1/*N* portfolio is 0.0079, but those of the sample-based mean–variance portfolio and the sample-based mean–variance portfolio with short sale constraints are only -2.9122 and 0.0024, respectively. The sample-based mean–variance portfolio has an extremely large turnover of 24.6812 compared to 0.0686 for the naive 1/*N* portfolio. Last, only the naive 1/*N* strategy has a positive value of MPPM (0.0640), while the other optimal strategies have a negative value of MPPM (-4.3921 for the sample-based mean–variance portfolio and -0.0791 for the sample-based mean–variance portfolio with short sale constraints).

Second, the short sale constraints improve performance, but not sufficiently. The sample-based mean–variance portfolio with short sale constraints consistently outperforms that without constraints. The short sale constraints seem to help reduce extreme weights that fluctuate substantially over time to a certain degree. However, the sample-based mean–variance portfolio with short sale constraints still underperforms the naive 1/*N* portfolio.

Third, the magnitude of the difference between the performance measures for the naive 1/*N* strategy and for the optimal portfolio strategies increases as the number of stocks in the portfolios increases. For example, in the case of *N* = 10, the differences in the Sharpe ratio, CEQ return, turnover, and MPPM between the naive 1/*N* strategy and the sample-based mean–variance portfolio are 0.1288, 1.1876, -8.3135, and 2.9902, respectively. As the number of stocks in the portfolios increases to *N* = 40, the differences also increase to 0.2135, 3.9855, -33.9899, and 5.0607, respectively. Especially for the sample-based mean–variance portfolio, the standard deviation of the portfolio return is an increasing function of the number of stocks in the portfolio, although it should be a decreasing function if it is a truly optimal portfolio. As DeMiguel et al. (2009b) point out, the larger number of stocks implies more parameters to be estimated and, therefore, more room for estimation error. Moreover, all other things being equal, the larger number of stocks in a portfolio makes naive diversification more effective relative to optimal diversification.

From analyzing tail risk measures, first, we find that the skewness for the naive $1/N$ strategy is more negative than that for the optimal portfolio strategies. In particular, in the case of $N = 30$, the skewness measure for the naive $1/N$ portfolio is -0.6108, while those for the sample-based mean–variance portfolio and the sample-based mean–variance portfolio with short sale constraints are -0.4804 and 0.0628, respectively. Moreover, as the number of stocks in the portfolios increases, the skewness for the naive $1/N$ portfolio decreases than the mean–variance portfolios. For example, the change in skewness from the naive $1/N$ portfolios with $N = 10$ to those with $N = 40$ is -0.4490, while those for the sample-based mean–variance portfolio and the sample-based mean–variance portfolio with short sale constraints are -0.2636 and -0.0305, respectively. This tendency in skewness for the naive $1/N$ portfolio implies that naive diversification increases tail risk.

Second, the kurtosis for the naive $1/N$ strategy is more positive than for the sample-based mean–variance portfolio with short sale constraints. Although the sample-based mean–variance portfolio has the largest value of the kurtosis among the strategies we consider, this is largely due to outliers resulting from extreme portfolio weights. Furthermore, we find that the kurtosis for the naive $1/N$ strategy is an increasing function of the number of stocks in the portfolios while that for the sample-based mean–variance portfolio with short sale constraints is a decreasing function of the number of portfolio stocks. Similar to the skewness results, this tendency in the kurtosis of the naive 1/*N* portfolio implies that naive diversification increases tail risk.

Third, the values of VaR and ES for the optimal portfolio strategies are higher than for the naive 1/*N* strategy. Moreover, while the VaR and ES for the sample-based mean–variance portfolio are increasing functions of the number of portfolio stocks, these measures for the naive 1/*N* portfolio are decreasing functions of the number of portfolio stocks. As for the results from the standard deviation (Panel B in Figure 2), this is due to the extreme weights that fluctuate substantially over time due to the estimation error of the optimal portfolio strategy.

4.2.2 Return distribution

[Insert Figure 3 about here]

Figure 3 shows the kernel smoothed histogram for the return distributions of portfolios consisting of randomly selected stocks from the naive 1/*N* strategy as well as the optimal portfolio strategies with $N = 5$ and $N = 50$. We normalize the portfolio return by using the mean and standard deviation of each portfolio. For this reason, the *x*-axis in Figure 3 is the mean deviation normalized by the standard deviation. From this normalization, we obtain 1,200,000 returns ($120 \times 10{,}000$) for each strategy. Figure 3 also reports the corresponding normal distribution generated by the pooled mean and pooled standard deviation from the normalized return distributions of the naive 1/*N* and optimal portfolio strategies.

Consistent with the results in Section 4.2.1, for both $N = 5$ and $N = 50$ Figure 3 shows that all strategies exhibit leptokurtic distribution, which means positive excess kurtosis and heavier tails than the normal distribution. However, the naive 1/*N* strategy has a more negatively skewed distribution than the optimal portfolio strategies. This means that the naive 1/*N* strategy has a larger and longer left tail distribution than the optimal portfolio strategies. In other words, the naive 1/*N* strategy tends to increase in left tail risk exposure and reduce upside potential. Moreover, this tendency gets stronger as the number of portfolio stocks increases. This result implies that the naive 1/*N* strategy may have a more concave payoff than the optimal portfolio strategies.

4.2.3 Concavity of portfolio payoff

[Insert Figure 4 about here]

Figure 4 presents the fraction of times that the portfolio exhibits a concave payoff.[14] Panels A and B report the results for Henriksson and Merton (1981) and Treynor and Mazuy (1966), respectively. In each panel, the upper figure shows the fraction of times the portfolio showed a significant $\gamma_2 < 0$ at the 5% level. The lower figure shows the fraction of times the portfolio showed a significant $\gamma_1 > 0$ and $\gamma_2 < 0$ at the 5% level. From the results of Henriksson and Merton (1981) and of Treynor and Mazuy (1966), we find that the fraction of times the portfolio exhibits a concave payoff in the naive 1/*N* strategy is much higher than for the optimal portfolio strategies. In particular, in the case of $N = 30$, from the results for Treynor and Mazuy (1966), 35.1% of the naive 1/*N* portfolio shows a concave payoff relative to the equity benchmark, while only 5.3% of the sample-based mean–variance portfolio and 15.1% of the

[14] Appendix 3 reports the tables for the fraction of times the portfolio exhibits a concave payoff.

sample-based mean–variance portfolio with short sale constraints show a concave payoff. Moreover, the fraction of times the portfolio shows a concave payoff in the naive 1/*N* strategy increases as the number of portfolio stocks increases. This implies that the 1/*N* diversification increases the concavity of the portfolio's payoff.

To summarize the results in Sections 4.1 and 4.2, using individual stocks as well as diversified equity portfolios, we find that the naive diversification strategy consistently outperforms optimal diversification strategies. In addition, naive diversification increases tail risk and makes portfolio returns more concave relative to an equity benchmark. Last, the tendency toward naive diversification increases as the number of portfolio stocks increases. Therefore, the outperformance of naive diversification relative to optimal diversification results from compensation for the increase in tail risk and the reduced upside potential associated with a concave payoff.

4.3 Results from robustness tests

In the benchmark case reported in Sections 4.1 and 4.2, we assume the following: (i) The length of the estimation window is M = 120 months, rather than M = {60, 180}; (ii) the holding period is one month rather than one year; (iii) the portfolio evaluated consists of only risky assets rather than also including risk-free assets; (iv) the 1/*N*-with-rebalancing strategy is used rather than the 1/*N*-buy-and-hold strategy; (v) in calculating CEQ, the investor has a risk aversion of $\gamma = 1$ rather than some other value, say $\gamma = \{3, 5, 10\}$; and (vi) in calculating the MPPM, the investor has a risk aversion of $\rho = 3$ rather than the value estimated from the market portfolio.[15]

[15] If the market portfolio has a lognormal return, $1+r_m$, then the risk aversion parameter ρ can be selected so that

To check whether our results are robust to these assumptions, we repeat the analysis conducted in Sections 4.1 and 4.2 after relaxing each of the assumptions aforementioned. In addition, since our major finding may be an artifact of the financial crisis and excessive tail risk, we conduct our analysis by using a sample that does not include the period of the US subprime crisis. Specifically, we use the sample that spans the period from January 1963 to December 2006. According to these robustness tests, the main empirical results are similar and qualitatively unchanged. To save space, we do not report the results from these robustness tests but they are available upon request.

5. Conclusion

This paper examines the outperformance of naive diversification relative to optimal diversification. This paper's main research question is why optimal diversification cannot outperform naive diversification. Contrary to the literature on optimal portfolio choice, we focus on the tail risk exposure of the 1/*N* strategy rather than the estimation error of the optimal strategy. To do this, we construct a portfolio for each strategy by using diversified equity portfolios or individual stocks. Then, we empirically compare naive diversification against optimal diversification in terms of performance, tail risk, return distribution, and portfolio return concavity.

Our paper's major findings can be summarized as follows. First, the naive diversification strategy outperforms the optimal diversification strategy. In general, the Sharpe ratio, CEQ return, and MPPM of the naive diversification strategy are higher or indistinguishable from those of the

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$$\rho = \frac{\ln[E(1+\tilde{r}_m)] - \ln(1+r_f)}{Var[\ln(1+\tilde{r}_m)]}.$$

optimal diversification strategy. Moreover, the naive diversification strategy has the lowest turnover. Second, naive diversification increases left tail risk exposure. The naive 1/*N* portfolio tends not only to have negative skewness and positive excess kurtosis but also to diminish the upside return potential. Third, naive diversification makes the portfolio returns more concave relative to the equity benchmark. From the test of Henriksson and Merton (1981) and Treynor and Mazuy (1966), we find that the portfolio return from naive diversification exhibits a much more concave payoff pattern than the optimal diversification. Fourth, these tendencies of naive diversification get stronger as the number of portfolio stocks increases.

We do not want to imply that naive diversification is a generally recommendable investment strategy. Instead, we imply that the outperformance of naive diversification relative to optimal diversification can be explained as due to compensation for the increase in tail risk and the reduced upside potential associated with a concave payoff. In addition, this paper suggests that to evaluate the performance of a particular asset allocation strategy, tail risk exposure should be taken into account in addition to portfolio risk.

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Appendix 1

List of various asset allocation models considered by DeMiguel et al. (2009b)

This table lists the various asset allocation models considered by DeMiguel et al. (2009b). The last column of the table gives the abbreviation used to refer to the strategy in Tables 1 and 2.

#	Model	Abbreviation
Naive		
1	Naive 1/*N*	1/*N*
Classical approach		
2	Sample-based mean–variance in sample	mv (in sample)
3	Sample-based mean–variance	mv
Bayesian approach		
4	Bayes–Stein	bs
5	Bayesian data-and-model	dm
Moment restrictions		
6	Minimum variance	min
7	MacKinlay and Pastor’s (2000) missing-factor model	mp
Portfolio constraints		
8	Sample-based mean–variance with short sale constraints	mv-c
9	Bayes–Stein with short sale constraints	bs-c
10	Minimum-variance with short sale constraints	min-c
11	Minimum-variance with generalized constraints	g-min-c
Optimal combinations of portfolios		
12	Kan and Zhou’s (2007) “three-fund” model	mv-min
13	Mixture of minimum variance and 1/*N*	ew-min

Appendix 2

Performance and tail risk measures for a portfolio consisting of randomly selected stocks, January 1963 to December 2011

This table reports the mean value of the performance and tail risk measures for portfolios consisting of randomly selected stocks from the naive 1/*N* and optimal portfolio strategies. Panels A to C report the results for the naive 1/*N* strategy, the sample-based mean–variance strategy, and the sample-based mean–variance strategy with short sale constraints, respectively. The performance and tail risk measures we consider are the mean, standard deviation, Sharpe ratio, CEQ return, turnover, MPPM (ρ = 3), skewness, kurtosis, historical VaR (confidence level = 95%), and historical ES (confidence level = 95%). The number of portfolio stocks, *N*, ranges from two to 50. For each *N*, *N* stocks are randomly selected *B* times to construct portfolios (in our study, *B* = 10,000). To obtain sensible measures of performance and tail risk for the portfolios from the time-series regressions, we require that all *N* stocks that are randomly selected to construct portfolios have at least 120 months of overlapping return history. We construct portfolios by using a rolling-sample approach. We choose an estimation window of length *M* = 120 months. To avoid oversampling returns in the period of the US subprime crisis, we generate the first 120 months of portfolio returns.

Appendix 2 (continued)

Panel A: Naive 1/*N* strategy										
N	Mean	Standard deviation	Sharpe ratio	CEQ	Turnover	MPPM	Skewness	Kurtosis	VaR	ES
2	0.0092	0.0844	0.1116	0.0052	0.0498	-0.0219	0.3854	5.4096	0.1144	0.1579
3	0.0093	0.0746	0.1270	0.0063	0.0582	0.0083	0.2365	5.4292	0.1012	0.1431
4	0.0093	0.0690	0.1367	0.0067	0.0620	0.0230	0.1335	5.4977	0.0933	0.1345
5	0.0093	0.0650	0.1454	0.0071	0.0638	0.0331	0.0563	5.5903	0.0879	0.1282
6	0.0093	0.0623	0.1518	0.0073	0.0652	0.0395	-0.0080	5.6916	0.0842	0.1242
7	0.0092	0.0599	0.1560	0.0073	0.0656	0.0433	-0.0832	5.7109	0.0810	0.1209
8	0.0092	0.0585	0.1598	0.0074	0.0662	0.0462	-0.1276	5.8210	0.0789	0.1188
9	0.0092	0.0573	0.1642	0.0076	0.0668	0.0495	-0.1787	5.8530	0.0774	0.1175
10	0.0093	0.0559	0.1686	0.0076	0.0670	0.0524	-0.2316	5.8426	0.0755	0.1152
11	0.0092	0.0550	0.1712	0.0077	0.0673	0.0541	-0.2679	5.9060	0.0742	0.1140
12	0.0092	0.0544	0.1714	0.0076	0.0675	0.0541	-0.2970	5.9847	0.0732	0.1133
13	0.0092	0.0537	0.1749	0.0077	0.0677	0.0561	-0.3233	5.9791	0.0723	0.1123
14	0.0092	0.0530	0.1767	0.0078	0.0679	0.0572	-0.3542	5.9926	0.0713	0.1111
15	0.0091	0.0525	0.1771	0.0077	0.0678	0.0570	-0.3963	6.0785	0.0705	0.1110
16	0.0091	0.0521	0.1792	0.0078	0.0680	0.0583	-0.4110	6.0839	0.0701	0.1103
17	0.0091	0.0516	0.1811	0.0078	0.0681	0.0591	-0.4318	6.1275	0.0694	0.1096
18	0.0092	0.0513	0.1824	0.0078	0.0683	0.0598	-0.4679	6.1706	0.0690	0.1093
19	0.0092	0.0510	0.1845	0.0079	0.0683	0.0610	-0.4680	6.1725	0.0685	0.1087
20	0.0091	0.0507	0.1836	0.0078	0.0686	0.0604	-0.4806	6.1940	0.0684	0.1084
22	0.0091	0.0503	0.1857	0.0078	0.0685	0.0614	-0.5195	6.2596	0.0675	0.1080
24	0.0091	0.0499	0.1873	0.0079	0.0686	0.0621	-0.5405	6.2892	0.0670	0.1075
26	0.0091	0.0493	0.1899	0.0079	0.0688	0.0632	-0.5741	6.3388	0.0662	0.1067
28	0.0092	0.0489	0.1913	0.0079	0.0686	0.0640	-0.5893	6.3814	0.0656	0.1060
30	0.0091	0.0486	0.1921	0.0079	0.0686	0.0640	-0.6108	6.3906	0.0653	0.1055
32	0.0091	0.0483	0.1928	0.0079	0.0688	0.0642	-0.6220	6.4168	0.0649	0.1052
34	0.0091	0.0482	0.1936	0.0079	0.0688	0.0646	-0.6443	6.4778	0.0646	0.1051
36	0.0091	0.0480	0.1940	0.0079	0.0688	0.0646	-0.6571	6.4982	0.0646	0.1050
38	0.0091	0.0479	0.1954	0.0080	0.0690	0.0654	-0.6651	6.5051	0.0643	0.1050
40	0.0091	0.0477	0.1956	0.0079	0.0688	0.0653	-0.6806	6.5690	0.0640	0.1046
45	0.0091	0.0473	0.1974	0.0080	0.0689	0.0661	-0.7049	6.6022	0.0634	0.1040
50	0.0091	0.0470	0.1985	0.0080	0.0689	0.0664	-0.7247	6.6299	0.0630	0.1037

Appendix 2 (continued)

Panel B: Sample-based mean–variance strategy										
N	Mean	Standard deviation	Sharpe ratio	CEQ	Turnover	MPPM	Skewness	Kurtosis	VaR	ES
2	0.0111	0.5177	0.0575	-0.8113	4.1511	-2.7841	0.0936	24.9496	0.2663	0.8576
3	0.0134	0.5744	0.0609	-1.3278	5.3772	-2.9336	0.0196	22.4049	0.2844	0.9218
4	0.0128	0.5217	0.0608	-1.1053	5.4474	-2.7787	-0.0239	19.9711	0.2822	0.8623
5	0.0075	0.5007	0.0579	-0.9971	5.8071	-2.8142	-0.0893	18.8855	0.2895	0.8836
6	0.0085	0.5366	0.0554	-1.4028	6.4725	-2.6486	-0.0946	18.0310	0.2930	0.9103
7	0.0071	0.5222	0.0508	-1.3328	6.7314	-2.7225	-0.1260	17.8087	0.2964	0.9091
8	0.0066	0.5625	0.0459	-1.4753	7.8809	-2.8994	-0.2302	18.2846	0.3121	0.9789
9	0.0057	0.5444	0.0435	-1.3013	8.3950	-2.9476	-0.2047	17.9634	0.3194	0.9657
10	0.0015	0.5162	0.0398	-1.1800	8.3805	-2.9379	-0.2450	16.9727	0.3210	0.9649
11	0.0070	0.5823	0.0365	-1.7613	9.7223	-3.0440	-0.2401	17.5052	0.3332	0.9915
12	-0.0011	0.5583	0.0311	-1.3593	10.1914	-3.1081	-0.3556	17.4608	0.3444	1.0468
13	-0.0027	0.5976	0.0284	-1.6332	10.9955	-3.1662	-0.4060	17.8731	0.3596	1.1266
14	-0.0008	0.6316	0.0264	-1.9044	11.7759	-3.3533	-0.3715	18.0787	0.3688	1.1600
15	-0.0035	0.6064	0.0226	-1.4921	12.6407	-3.5081	-0.3416	17.7134	0.3743	1.1561
16	-0.0015	0.6677	0.0207	-2.2208	13.5785	-3.5520	-0.3732	18.0583	0.3803	1.2077
17	-0.0076	0.6347	0.0169	-1.6208	13.9261	-3.5340	-0.4470	18.3084	0.3903	1.2501
18	-0.0037	0.6457	0.0156	-1.9025	14.2317	-3.5022	-0.3970	18.1988	0.3878	1.2001
19	-0.0001	0.6824	0.0138	-2.4102	15.1230	-3.6225	-0.3645	18.2015	0.3968	1.2160
20	-0.0086	0.6563	0.0107	-1.8260	15.9083	-3.6458	-0.4057	17.9663	0.4047	1.2906
22	-0.0070	0.6932	0.0081	-2.2640	17.2180	-3.8679	-0.3958	18.4341	0.4224	1.3188
24	-0.0076	0.7095	0.0034	-2.0955	19.8306	-4.1218	-0.4380	18.4722	0.4417	1.3560
26	-0.0082	0.7933	0.0010	-3.1504	20.8574	-4.0529	-0.4561	18.9705	0.4529	1.4785
28	-0.0102	0.7882	-0.0047	-2.7760	22.4218	-4.1731	-0.5441	19.5072	0.4711	1.4987
30	-0.0113	0.8223	-0.0079	-2.9122	24.6812	-4.3921	-0.4804	19.3595	0.4933	1.5693
32	-0.0049	0.8409	-0.0076	-3.0447	25.9826	-4.4098	-0.4302	19.7483	0.4878	1.5224
34	-0.0112	0.8918	-0.0118	-3.4961	28.2495	-4.5967	-0.5030	20.2150	0.5082	1.6736
36	-0.0141	0.8822	-0.0131	-3.3035	29.3153	-4.7934	-0.4722	20.1709	0.5215	1.6900
38	-0.0094	0.9631	-0.0141	-3.9653	31.9517	-4.8758	-0.4603	20.7779	0.5287	1.7546
40	-0.0132	0.9719	-0.0179	-3.9775	34.0587	-4.9954	-0.5086	21.4398	0.5471	1.8182
45	-0.0138	1.0906	-0.0207	-4.7457	39.5816	-5.4509	-0.5186	22.6987	0.5875	2.0069
50	-0.0144	1.1801	-0.0223	-5.6389	43.9719	-5.5398	-0.4514	23.8790	0.6113	2.1424

Appendix 2 (continued)

Panel C: Sample-based mean–variance strategy with short sale constraints

N	Mean	Standard deviation	Sharpe ratio	CEQ	Turnover	MPPM	Skewness	Kurtosis	VaR	ES
2	0.0077	0.0892	0.0863	0.0034	0.0678	-0.0590	0.2728	5.1567	0.1273	0.1758
3	0.0075	0.0868	0.0854	0.0035	0.1017	-0.0538	0.1700	5.0858	0.1256	0.1749
4	0.0073	0.0868	0.0826	0.0033	0.1220	-0.0561	0.1477	5.1372	0.1264	0.1762
5	0.0071	0.0865	0.0812	0.0032	0.1361	-0.0562	0.1289	5.1240	0.1266	0.1764
6	0.0071	0.0869	0.0793	0.0031	0.1472	-0.0588	0.1148	5.1429	0.1273	0.1779
7	0.0070	0.0872	0.0785	0.0030	0.1541	-0.0606	0.1023	5.0661	0.1279	0.1789
8	0.0070	0.0876	0.0777	0.0030	0.1599	-0.0619	0.0952	5.1135	0.1290	0.1800
9	0.0071	0.0883	0.0783	0.0030	0.1652	-0.0634	0.0896	5.0559	0.1303	0.1816
10	0.0070	0.0882	0.0774	0.0029	0.1702	-0.0638	0.0846	5.0294	0.1302	0.1813
11	0.0070	0.0891	0.0762	0.0028	0.1733	-0.0670	0.0798	5.0056	0.1317	0.1835
12	0.0069	0.0892	0.0749	0.0027	0.1777	-0.0685	0.0891	5.0728	0.1319	0.1836
13	0.0070	0.0893	0.0757	0.0028	0.1798	-0.0678	0.0869	4.9787	0.1324	0.1838
14	0.0069	0.0897	0.0751	0.0027	0.1832	-0.0694	0.0850	4.9864	0.1329	0.1847
15	0.0069	0.0899	0.0746	0.0026	0.1850	-0.0711	0.0791	4.9790	0.1331	0.1854
16	0.0069	0.0901	0.0752	0.0027	0.1871	-0.0707	0.0696	4.9773	0.1337	0.1859
17	0.0067	0.0902	0.0727	0.0025	0.1898	-0.0731	0.0729	4.9606	0.1339	0.1861
18	0.0069	0.0906	0.0740	0.0026	0.1907	-0.0734	0.0636	4.9545	0.1346	0.1874
19	0.0070	0.0907	0.0749	0.0027	0.1921	-0.0722	0.0742	4.9176	0.1345	0.1868
20	0.0068	0.0909	0.0724	0.0025	0.1951	-0.0753	0.0722	4.9425	0.1351	0.1875
22	0.0068	0.0910	0.0729	0.0025	0.1981	-0.0753	0.0706	4.9076	0.1354	0.1876
24	0.0067	0.0916	0.0714	0.0023	0.1988	-0.0788	0.0689	4.8814	0.1370	0.1893
26	0.0068	0.0916	0.0725	0.0025	0.2035	-0.0771	0.0681	4.9227	0.1365	0.1892
28	0.0069	0.0920	0.0725	0.0024	0.2036	-0.0781	0.0636	4.8766	0.1369	0.1900
30	0.0068	0.0919	0.0717	0.0024	0.2072	-0.0791	0.0628	4.8626	0.1372	0.1900
32	0.0068	0.0920	0.0726	0.0024	0.2082	-0.0787	0.0715	4.8567	0.1375	0.1901
34	0.0067	0.0922	0.0707	0.0022	0.2104	-0.0813	0.0614	4.8277	0.1377	0.1907
36	0.0068	0.0924	0.0718	0.0023	0.2115	-0.0805	0.0617	4.8233	0.1379	0.1910
38	0.0068	0.0930	0.0719	0.0023	0.2131	-0.0825	0.0603	4.7679	0.1390	0.1922
40	0.0067	0.0924	0.0716	0.0023	0.2152	-0.0810	0.0541	4.7799	0.1383	0.1913
45	0.0068	0.0931	0.0721	0.0023	0.2178	-0.0826	0.0552	4.7942	0.1394	0.1926
50	0.0067	0.0931	0.0704	0.0022	0.2219	-0.0841	0.0585	4.7617	0.1394	0.1926

Appendix 3

Fraction of times the portfolio exhibited a concave payoff as a function of the number of portfolio stocks, January 1963 to December 2011

This table reports the fraction of times the portfolio showed a concave payoff. To examine the concavity of a portfolio's payoff, we use the coefficients of Henriksson and Merton (1981) and Treynor and Mazuy (1966). These coefficients are computed from the regression $r_t = \gamma_0 + \gamma_1 m_t + \gamma_2 c_t + \varepsilon_t$, where r_t and m_t are the excess returns on the portfolio and the market, respectively. The timing variable is c_t = Max($-m_t$, 0) for Henriksson and Merton (1981) and $c_t = m_t^2$ for Treynor and Mazuy (1966). Panels A to C report the results for the naive 1/*N* strategy, the sample-based mean–variance strategy, and the sample-based mean–variance strategy with short sale constraints, respectively. Each panel reports the fraction of times that the portfolio showed a significant $\gamma_2 < 0$ at the 5% level and the fraction of times that the portfolio showed a significant $\gamma_1 > 0$ and $\gamma_2 < 0$ at the 5% level using the coefficients of Henriksson and Merton (1981) or Treynor and Mazuy (1966). The number of portfolio stocks, *N*, ranges from two to 50. For each *N*, *N* stocks are randomly selected *B* times to construct portfolios (in our study, *B* = 10,000). To obtain sensible measures of performance and tail risk for the portfolios from the time-series regressions, we require that all *N* stocks that are randomly selected to construct a portfolio have at least 120 months of overlapping return history. We construct portfolios by using a rolling-sample approach. We choose an estimation window of length *M* = 120 months. To avoid oversampling returns in the period of the US subprime crisis, we generate the first 120 months of portfolio returns.

Appendix 3 (continued)

Panel A: Naive 1/*N* strategy				
	Henriksson and Merton (1981)		Treynor and Mazuy (1966)	
N	Freq (γ_2<0 sig.) (%)	Freq (γ_1>0, γ_2<0 sig.) (%)	Freq (γ_2<0 sig.) (%)	Freq (γ_1>0, γ_2<0 sig.) (%)
2	0.1101	0.0615	0.1377	0.1318
3	0.1286	0.0901	0.1583	0.1567
4	0.1314	0.1020	0.1678	0.1665
5	0.1474	0.1194	0.1803	0.1795
6	0.1521	0.1307	0.1952	0.1947
7	0.1571	0.1371	0.2047	0.2047
8	0.1686	0.1544	0.2138	0.2138
9	0.1727	0.1579	0.2252	0.2252
10	0.1836	0.1707	0.2338	0.2338
11	0.1878	0.1781	0.2419	0.2419
12	0.1874	0.1797	0.2420	0.2420
13	0.1940	0.1888	0.2531	0.2531
14	0.2045	0.1986	0.2644	0.2644
15	0.2111	0.2063	0.2747	0.2747
16	0.2138	0.2087	0.2796	0.2796
17	0.2181	0.2148	0.2845	0.2845
18	0.2315	0.2291	0.3005	0.3005
19	0.2292	0.2272	0.2998	0.2998
20	0.2279	0.2263	0.2976	0.2976
22	0.2384	0.2373	0.3125	0.3125
24	0.2437	0.2430	0.3200	0.3200
26	0.2553	0.2547	0.3354	0.3354
28	0.2660	0.2656	0.3464	0.3464
30	0.2683	0.2681	0.3508	0.3508
32	0.2799	0.2798	0.3630	0.3630
34	0.2810	0.2810	0.3744	0.3744
36	0.2857	0.2856	0.3678	0.3678
38	0.2876	0.2876	0.3759	0.3759
40	0.2929	0.2929	0.3842	0.3842
45	0.3066	0.3066	0.3996	0.3996
50	0.3133	0.3133	0.4142	0.4142

Appendix 3 (continued)

Panel B: Sample-based mean–variance strategy				
	Henriksson and Merton (1981)		Treynor and Mazuy (1966)	
N	Freq (γ_2<0 sig.) (%)	Freq (γ_1>0, γ_2<0 sig.) (%)	Freq (γ_2<0 sig.) (%)	Freq (γ_1>0, γ_2<0 sig.) (%)
2	0.1163	0.0249	0.1284	0.0940
3	0.1109	0.0281	0.1242	0.0945
4	0.1095	0.0284	0.1203	0.0913
5	0.1138	0.0249	0.1218	0.0919
6	0.1133	0.0253	0.1234	0.0938
7	0.1201	0.0215	0.1221	0.0883
8	0.1219	0.0187	0.1347	0.0980
9	0.1248	0.0174	0.1309	0.0938
10	0.1223	0.0138	0.1274	0.0928
11	0.1210	0.0122	0.1278	0.0900
12	0.1253	0.0110	0.1343	0.0903
13	0.1216	0.0087	0.1257	0.0831
14	0.1258	0.0078	0.1330	0.0902
15	0.1236	0.0066	0.1281	0.0831
16	0.1278	0.0060	0.1332	0.0791
17	0.1325	0.0055	0.1352	0.0790
18	0.1228	0.0049	0.1324	0.0754
19	0.1261	0.0024	0.1321	0.0722
20	0.1243	0.0022	0.1258	0.0702
22	0.1289	0.0017	0.1272	0.0615
24	0.1311	0.0013	0.1339	0.0648
26	0.1231	0.0004	0.1256	0.0577
28	0.1241	0.0003	0.1279	0.0537
30	0.1210	0.0001	0.1293	0.0528
32	0.1194	0.0002	0.1249	0.0477
34	0.1269	0.0001	0.1275	0.0419
36	0.1271	0.0000	0.1257	0.0391
38	0.1207	0.0001	0.1197	0.0346
40	0.1198	0.0000	0.1198	0.0327
45	0.1089	0.0000	0.1105	0.0259
50	0.1019	0.0000	0.1024	0.0228

Appendix 3 (continued)

Panel C: Sample-based mean–variance strategy with short sale constraints				
	Henriksson and Merton (1981)		Treynor and Mazuy (1966)	
N	Freq (γ_2<0 sig.) (%)	Freq (γ_1>0, γ_2<0 sig.) (%)	Freq (γ_2<0 sig.) (%)	Freq (γ_1>0, γ_2<0 sig.) (%)
2	0.1201	0.0578	0.1437	0.1372
3	0.1236	0.0694	0.1534	0.1500
4	0.1218	0.0729	0.1496	0.1470
5	0.1216	0.0725	0.1539	0.1511
6	0.1282	0.0803	0.1615	0.1588
7	0.1237	0.0826	0.1504	0.1489
8	0.1308	0.0888	0.1636	0.1617
9	0.1353	0.0899	0.1599	0.1588
10	0.1267	0.0891	0.1553	0.1548
11	0.1264	0.0889	0.1547	0.1536
12	0.1268	0.0896	0.1600	0.1589
13	0.1275	0.0935	0.1536	0.1528
14	0.1266	0.0887	0.1564	0.1554
15	0.1253	0.0929	0.1494	0.1486
16	0.1323	0.0983	0.1591	0.1586
17	0.1266	0.0944	0.1571	0.1569
18	0.1263	0.0902	0.1585	0.1577
19	0.1203	0.0900	0.1538	0.1534
20	0.1259	0.0953	0.1549	0.1544
22	0.1193	0.0912	0.1517	0.1511
24	0.1168	0.0890	0.1462	0.1462
26	0.1241	0.0956	0.1518	0.1515
28	0.1303	0.1022	0.1572	0.1570
30	0.1212	0.0959	0.1515	0.1514
32	0.1249	0.0982	0.1498	0.1493
34	0.1267	0.0983	0.1546	0.1543
36	0.1193	0.0969	0.1415	0.1414
38	0.1157	0.0954	0.1446	0.1445
40	0.1164	0.0936	0.1485	0.1484
45	0.1156	0.0950	0.1444	0.1444
50	0.1115	0.0924	0.1445	0.1444

Table 1

Extended results of DeMiguel et al. (2009b): Performance and tail risk measures for a portfolio consisting of diversified equity portfolios, January 1963 to December 2011

This table reports the performance and tail risk measures for a portfolio consisting of diversified equity portfolios across the asset allocation models listed in Appendix 1. The dataset we consider for the diversified equity portfolios comprises the portfolios from the Fama–French four-factor model (the FF-4-factor dataset of DeMiguel et al., 2009b). Specifically, the dataset consists of 20 size- and book-to-market portfolios and the MKT, SMB, HML, and UMD portfolios. The number of diversified equity portfolios in the dataset is 24. Following Wang (2005) and DeMiguel et al. (2009b), of the 25 size- and book-to-market portfolios, we exclude the five portfolios containing the largest firms because the MKT, SMB, and HML portfolios are almost a linear combination of the 25 Fama–French portfolios. This dataset is collected from Ken French's website. The performance and tail risk measures we consider are the mean, standard deviation, Sharpe ratio, CEQ return, turnover, skewness, kurtosis, historical VaR (confidence level = 95%), and historical ES (confidence level = 95%). We use all 24 assets in the dataset to construct the portfolios. We construct the portfolios by using a rolling-sample approach, except for the sample-based mean–variance in the sample (mv - in sample). We choose an estimation window of length M = 120 months.

Strategy	Mean	Standard deviation	Sharpe ratio	CEQ	Turnover	Skewness	Kurtosis	VaR	ES
1/*N*	0.0068	0.0488	0.1393	0.0056	0.0197	-0.6051	5.9441	0.0695	0.1124
mv (in sample)	0.1008	0.2012	0.5008	0.0805	-	0.1129	4.8421	0.1767	0.3408
mv	0.0780	1.0123	0.0770	-0.4344	145.4396	19.9633	420.0406	0.0742	0.4063
bs	0.0525	0.6604	0.0795	-0.1656	46.7942	19.8739	417.4976	0.0481	0.2709
dm	0.0518	0.4219	0.1227	-0.0372	59.8727	6.1453	102.3050	0.0961	0.5210
min	0.0001	0.0043	0.0124	0.0000	0.1336	0.1450	8.9281	0.0063	0.0103
mp	0.0056	0.0574	0.0972	0.0039	0.0358	-0.8088	5.5972	0.0842	0.1380
mv-c	0.0065	0.0458	0.1421	0.0055	0.2748	-0.7920	7.0535	0.0727	0.1100
bs-c	0.0066	0.0422	0.1571	0.0057	0.2315	-0.9402	8.1606	0.0679	0.1033
min-c	0.0045	0.0157	0.2887	0.0044	0.0324	-0.6739	6.6649	0.0211	0.0371
g-min-c	0.0060	0.0247	0.2417	0.0057	0.0330	-0.7985	7.2549	0.0331	0.0593
mv-min	0.0430	0.5232	0.0821	-0.0939	43.4867	19.7412	413.6792	0.0413	0.2197
ew-min	0.0001	0.0044	0.0158	0.0001	0.1331	0.1570	9.0905	0.0063	0.0103

Table 2

Timing coefficients for a portfolio consisting of diversified equity portfolios, January 1963 to December 2011

This table reports the timing coefficients for a portfolio consisting of diversified equity portfolios across the asset allocation models listed in Appendix 1. The diversified equity portfolios dataset we consider is the set of portfolios from the Fama–French four-factor model (the FF-4 factor dataset in DeMiguel et al., 2009b). Specifically, the dataset consists of 20 size- and book-to-market portfolios and the MKT, SMB, HML, and UMD portfolios. The number of diversified equity portfolios in the dataset is 24. Following Wang (2005) and DeMiguel et al. (2009b), of the 25 size- and book-to-market portfolios, we exclude the five portfolios containing the largest firms, because the MKT, SMB, and HML portfolios are almost a linear combination of the 25 Fama–French portfolios. This dataset is collected from Ken French's website. To examine the portfolio payoff concavity, we use the coefficients of Henriksson and Merton (1981) and Treynor and Mazuy (1966). The coefficients are computed from the regression $r_t = \gamma_0 + \gamma_1 m_t + \gamma_2 c_t + \varepsilon_t$, where r_t and m_t are the excess returns on the portfolio and the market, respectively. The timing variable is c_t = Max($-m_t$, 0) for Henriksson and Merton (1981) and $c_t = m_t^2$ for Treynor and Mazuy (1966). We use all 24 assets in the dataset to construct portfolios. We construct portfolios by using a rolling-sample approach, except for the sample-based mean–variance in sample (mv - in sample). We choose an estimation window of length M = 120 months. The t-statistics are given in parentheses. Numbers in bold denote the coefficients' statistical significance.

Strategy	Henriksson and Merton (1981)			Treynor and Mazuy (1966)		
	γ_0	γ_1	γ_2	γ_0	γ_1	γ_2
1/*N*	**0.0057**	**0.8568**	**-0.1660**	**0.0041**	**0.9337**	**-0.6007**
	(4.00)	(23.26)	(-2.76)	(3.92)	(47.85)	(-2.82)
mv (in sample)	**0.1069**	0.5138	-0.5195	**0.1039**	**0.7386**	-2.8301
	(7.40)	(1.38)	(-0.86)	(9.94)	(3.75)	(-1.32)
mv	0.0067	**4.0512**	3.3301	0.0530	**2.4161**	6.4085
	(0.09)	(2.14)	(1.08)	(0.99)	(2.41)	(0.59)
bs	0.0071	**2.6170**	2.1073	0.0373	**1.5763**	3.6992
	(0.15)	(2.12)	(1.05)	(1.07)	(2.41)	(0.52)
dm	**0.0623**	0.1132	-0.6820	**0.0597**	0.3989	-4.2791
	(2.02)	(0.14)	(-0.53)	(2.68)	(0.95)	(-0.94)
min	0.0005	0.0046	**-0.0309**	0.0002	**0.0190**	**-0.1028**
	(1.73)	(0.58)	(-2.39)	(0.91)	(4.54)	(-2.25)
mp	**0.0111**	**0.7298**	**-0.5410**	**0.0066**	**0.9741**	**-2.3335**
	(4.96)	(12.70)	(-5.77)	(4.14)	(32.56)	(-7.16)
mv-c	**0.0118**	**0.3892**	**-0.4329**	**0.0083**	**0.5833**	**-1.9421**
	(4.63)	(5.95)	(-4.05)	(4.58)	(17.01)	(-5.20)
bs-c	**0.0122**	**0.3228**	**-0.4308**	**0.0087**	**0.5164**	**-1.9126**
	(5.09)	(5.25)	(-4.29)	(5.08)	(16.04)	(-5.45)

min-c	**0.0080**	0.0035	**-0.2194**	**0.0057**	**0.1062**	**-0.7234**
	(7.71)	(0.13)	(-5.02)	(7.51)	(7.46)	(-4.66)
g-min-c	**0.0080**	**0.2740**	**-0.1987**	**0.0058**	**0.3676**	**-0.6214**
	(6.50)	(8.70)	(-3.86)	(6.47)	(21.94)	(-3.40)
mv-min	0.0081	**2.0498**	1.6089	0.0320	**1.2487**	2.4297
	(0.21)	(2.10)	(1.01)	(1.16)	(2.41)	(0.43)
ew-min	0.0006	0.0066	**-0.0318**	0.0002	**0.0214**	**-0.1066**
	(1.80)	(0.83)	(-2.47)	(0.97)	(5.12)	(-2.34)

Figure 1

Return distribution of a portfolio of diversified stock portfolios

This figure shows the kernel smoothed histogram for the return distributions of portfolios consisting of diversified stock portfolios from the naive 1/*N* and optimal portfolio strategies. This figure also shows the corresponding normal distribution generated by the pooled mean and pooled standard deviation from the return distributions of the naive 1/*N* and optimal portfolio strategies. The dataset we consider the diversified stock portfolios is the set of portfolios from the Fama–French four-factor model. Specifically, the dataset consists of 20 size- and book-to-market portfolios and the MKT, SMB, HML, and UMD portfolios. The number of diversified assets in the dataset is 24. Following Wang (2005) and DeMiguel et al. (2009b), of the 25 size- and book-to-market portfolios, we exclude the five portfolios containing the largest firms, because the MKT, SMB, and HML portfolios are almost a linear combination of the 25 Fama–French portfolios. This dataset is collected from Ken French's website. We use all 24 assets in the dataset to construct portfolios. We construct portfolios by using a rolling-sample approach. We choose an estimation window of length M = 120 months.

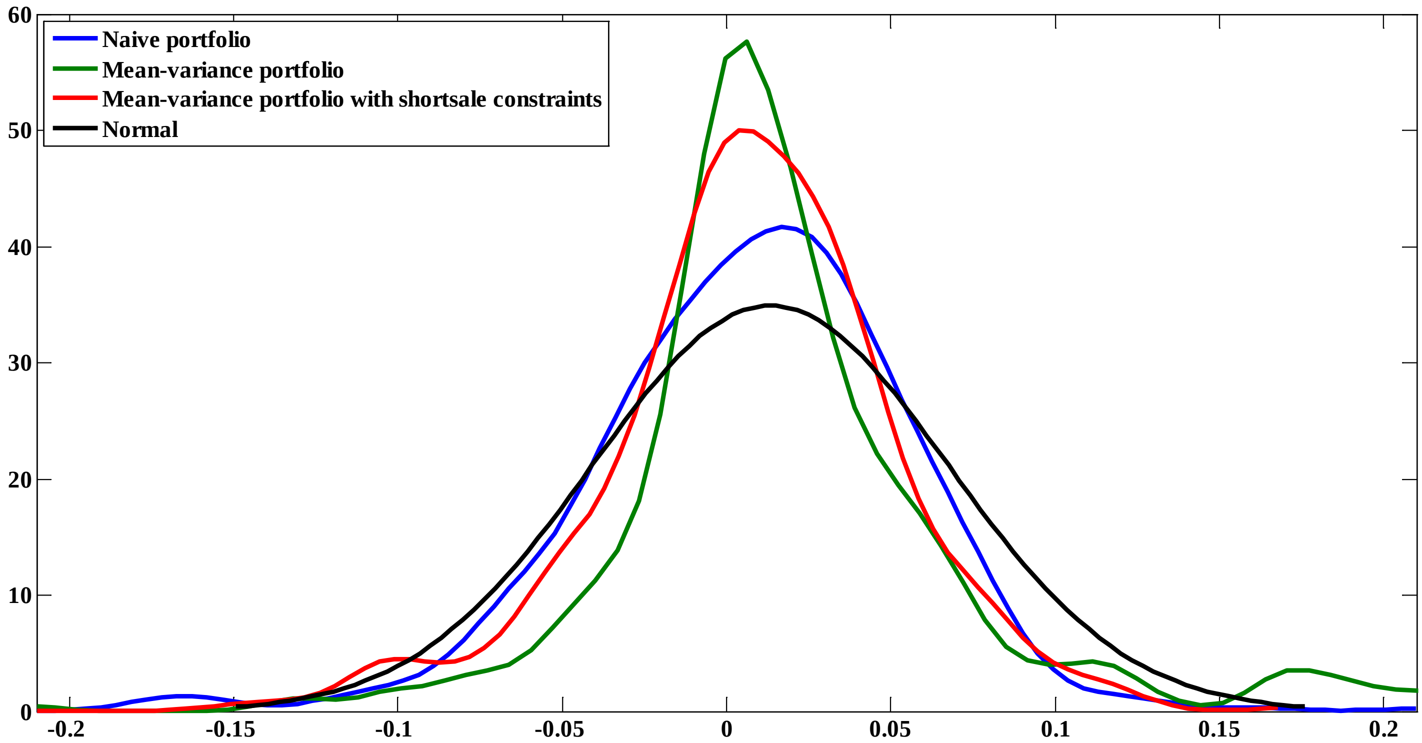

Naive portfolio
Mean-variance portfolio
Mean-variance portfolio with shortsale constraints
Normal
60
50
40
30
20
10
0
-0.2
-0.15
-0.1
-0.05
0
0.05
0.1
0.15
0.2

Figure 2

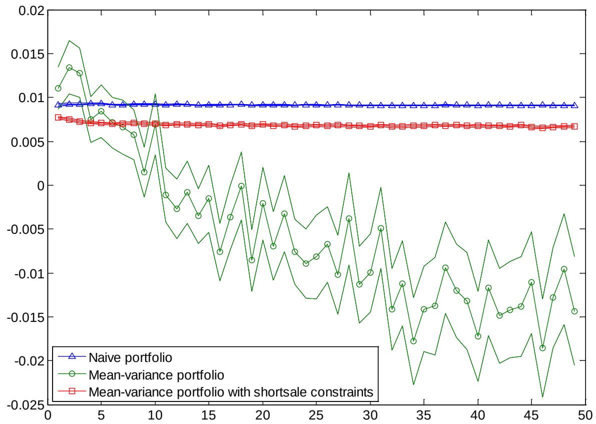
Performance and tail risk measures of a portfolio consisting of randomly selected stocks as a function of the number of portfolio stocks, January 1963 to December 2011

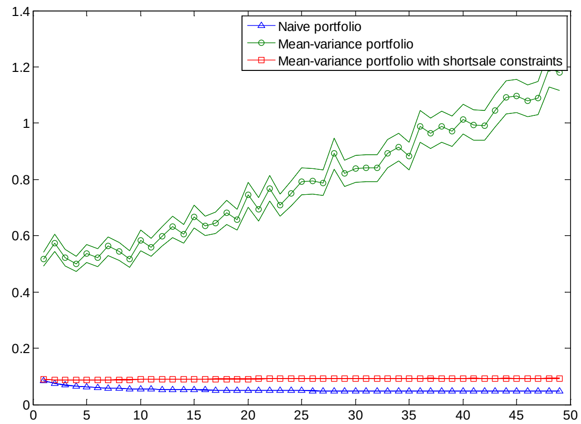
This figure shows the mean value of the performance and tail risk measures for portfolios consisting of randomly selected stocks from the naive 1/*N* and optimal portfolio strategies. The performance and tail risk measures we consider are the mean, standard deviation, Sharpe ratio, CEQ return, turnover, MPPM (ρ = 3), skewness, kurtosis, historical VaR (confidence level = 95%), and historical ES (confidence level = 95%). The number of stocks in the portfolio, *N*, ranges from two to 50. For each *N*, *N* stocks are randomly selected *B* times to construct portfolios (in our study, *B* = 10,000). To obtain sensible measures of performance and tail risk for portfolios from the time-series regressions, we require that all *N* stocks that are randomly selected to construct portfolios have at least 120 months of overlapping return history. We construct portfolios by using a rolling-sample approach. We choose an estimation window of length *M* = 120 months. To avoid oversampling returns in the period of the US subprime crisis, we generate the first 120 months of portfolio returns. The dotted line in the figure shows the confidence band of each measure. The range of the confidence band is between -2 and 2 standard deviations.

Figure 2 (continued)

Panel A. Mean

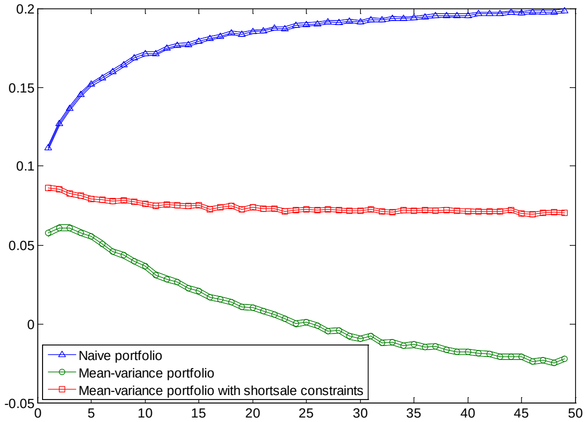
Panel B. Standard deviation

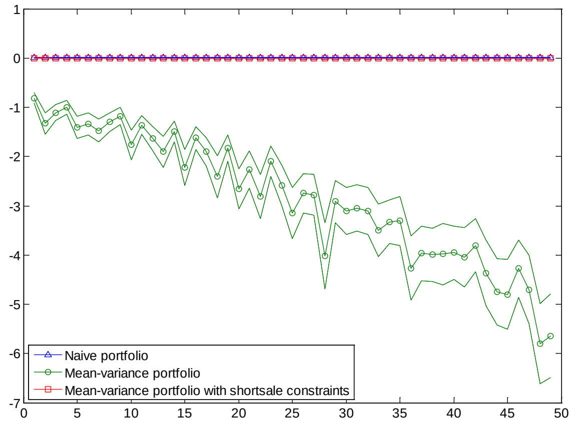




Panel C. Sharpe ratio

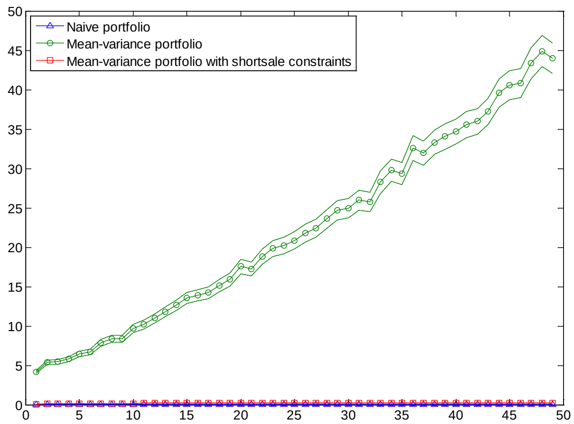
Panel D. CEQ return





Panel E. Turnover

Panel F. MPPM


Naive portfolio
Mean-variance portfolio
Mean-variance portfolio with shortsale constraints

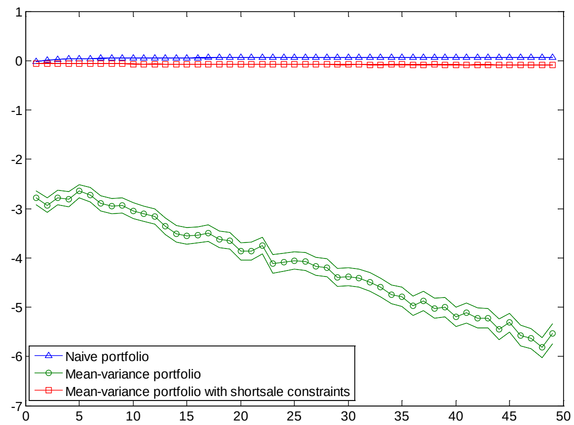
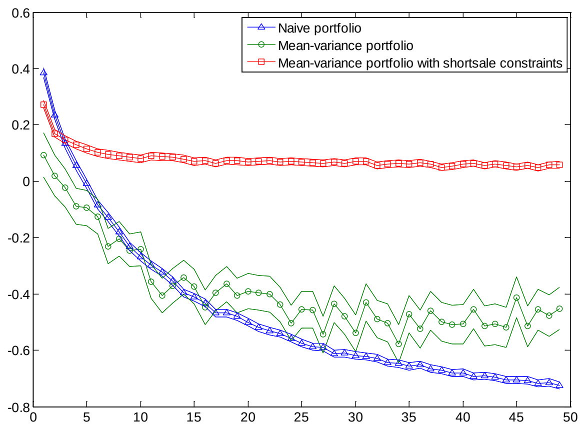
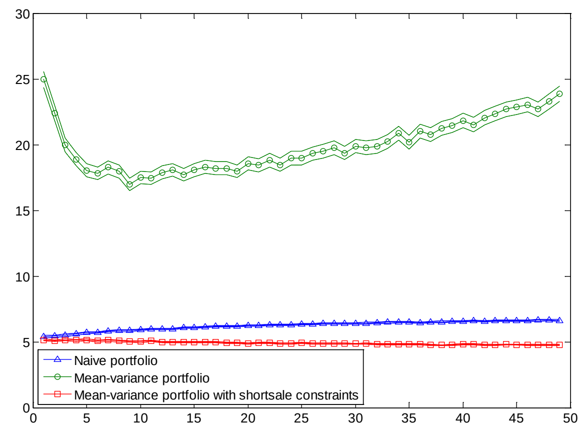

Naive portfolio
Mean-variance portfolio
Mean-variance portfolio with shortsale constraints

Figure 2 (continued)

Panel G. Skewness

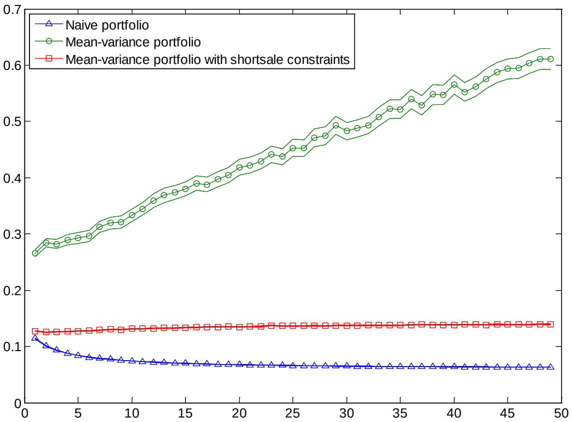
Panel H. Kurtosis





Panel I. VaR

Panel J. ES


Naive portfolio
Mean-variance portfolio
Mean-variance portfolio with shortsale constraints
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
5
10
15
20
25
30
35
40
45
50

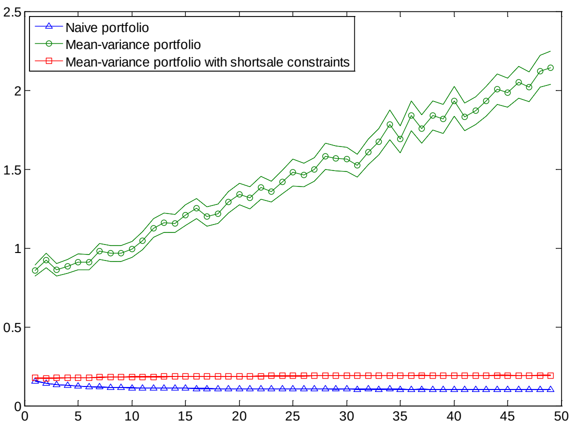

Naive portfolio
Mean-variance portfolio
Mean-variance portfolio with shortsale constraints
2.5
2
1.5
1
0.5
0
0
5
10
15
20
25
30
35
40
45
50

Figure 3

Return distribution of a portfolio of randomly selected stocks, January 1963 to December 2011

This figure shows the kernel smoothed histogram for the return distributions of portfolios consisting of randomly selected stocks from the naive 1/*N* and optimal portfolio strategies in case of *N* = 5 (P5) and *N* = 50 (P50). Here, 1/n, mv, and mvc represent the naive 1/*N* portfolio, the sample-based mean–variance portfolio, and the sample-based mean–variance portfolio with short sale constraints, respectively. For each *N*, *N* stocks are randomly selected *B* times to construct portfolios (in our study, *B* = 10,000). To obtain sensible measures of performance and tail risk for portfolios from the time-series regressions, we require that all *N* stocks that are randomly selected to construct portfolios have at least 120 months of overlapping return history. We construct portfolios by using a rolling-sample approach. We choose an estimation window of length *M* = 120 months. To avoid oversampling returns in the period of the US subprime crisis, we generate the first 120 months of portfolio returns. We normalize the portfolio return by using the mean and standard deviation of each portfolio. For this reason, the *x*-axis in this figure is the mean deviation normalized by the standard deviation. From this normalization, we obtain 1,200,000 returns (120 x 10,000) for each strategy. This figure also shows the corresponding normal distributions (P5-normal and P50-normal) generated, respectively, by the pooled mean and pooled standard deviation from the normalized return distributions of the naive 1/*N* and optimal portfolio strategies.

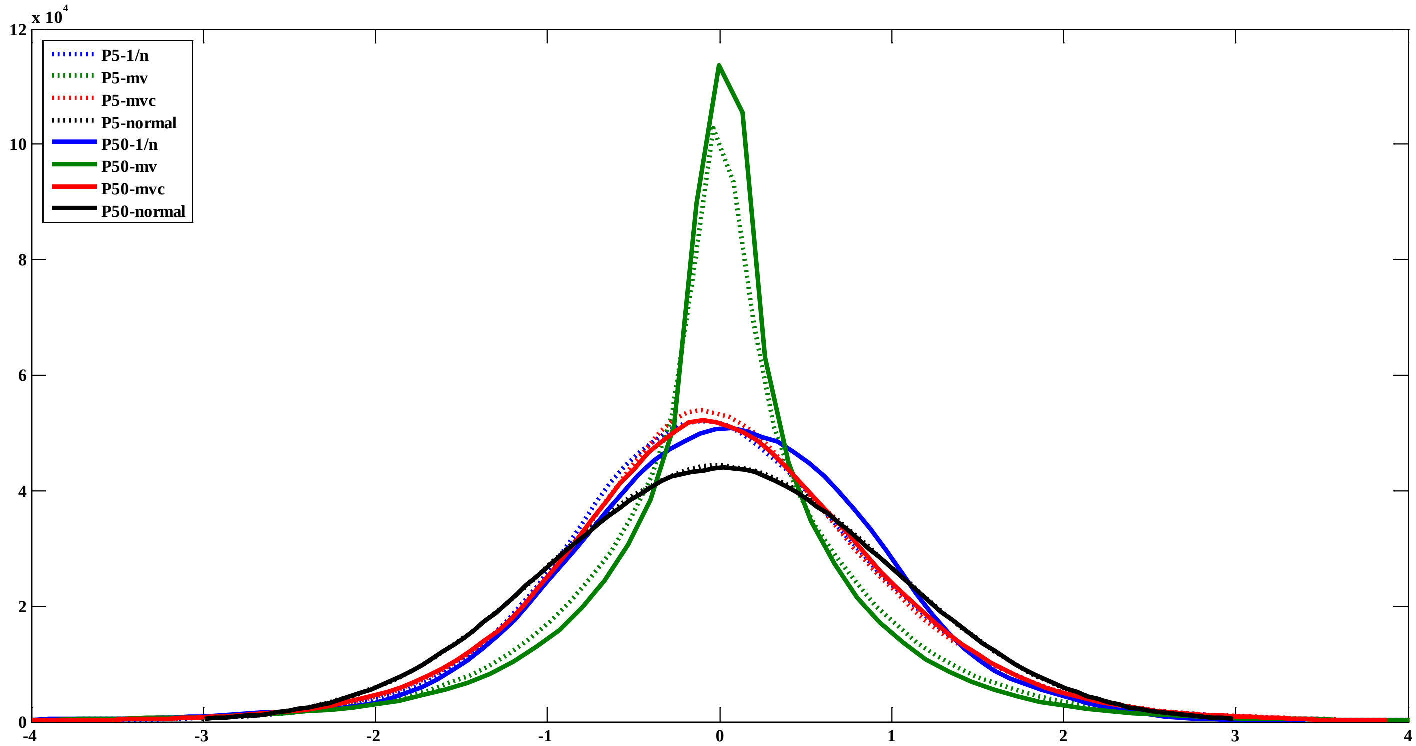

x 10^4
12
10
8
6
4
2
0
-4
-3
-2
-1
0
1
2
3
4
P5-1/n
P5-mv
P5-mvc
P5-normal
P50-1/n
P50-mv
P50-mvc
P50-normal

Figure 4

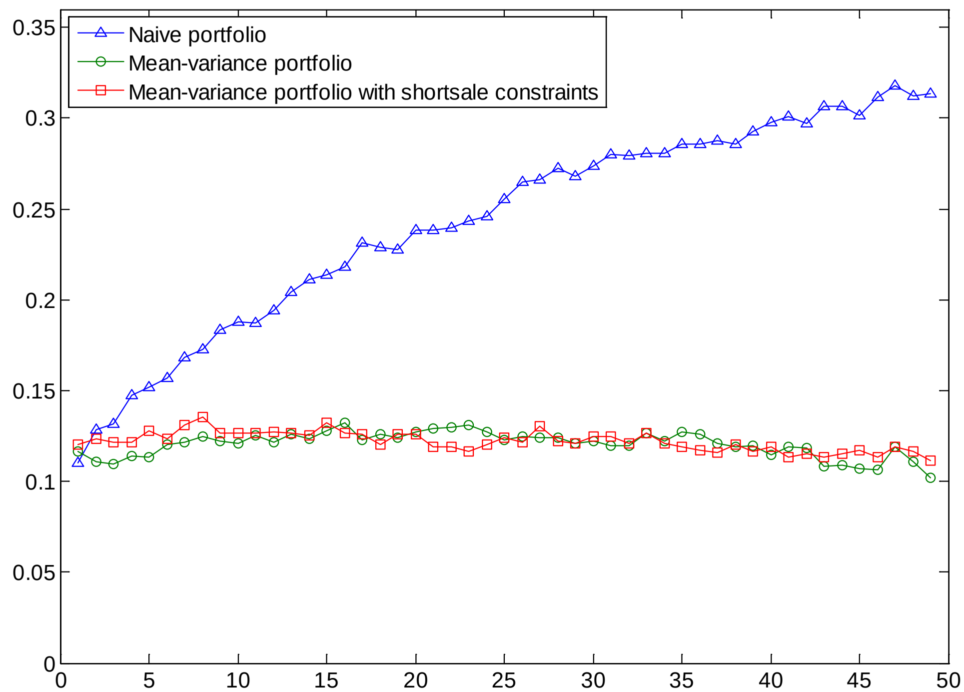
Fraction of times that the portfolio exhibits a concave payoff as a function of the number of portfolio stocks, January 1963 to December 2011

This figure presents the fraction of times that the portfolio shows a concave payoff. To examine the portfolio payoff concavity, we use the coefficients of Henriksson and Merton (1981) and Treynor and Mazuy (1966). The coefficients are computed from the regression $r_t = \gamma_0 + \gamma_1 m_t + \gamma_2 c_t + \varepsilon_t$, where r_t and m_t are the excess returns on the portfolio and the market, respectively. The timing variable is c_t = Max(-m_t, 0) for Henriksson and Merton (1981) and $c_t = m_t^2$ for Treynor and Mazuy (1966). Panels A and B report the results for Henriksson and Merton (1981) and Treynor and Mazuy (1966), respectively. In each panel, the upper figure shows the fraction of times that the portfolio shows a significant $\gamma_2 < 0$ at the 5% level. The lower figure shows the fraction of times that the portfolio shows a significant $\gamma_1 > 0$ and $\gamma_2 < 0$ at the 5% level. The number of stocks in the portfolio, *N*, ranges from two to 50. For each *N*, *N* stocks are randomly selected *B* times to construct portfolios (in our study, *B* = 10,000). To obtain sensible measures of performance and tail risk for the portfolios from the time-series regressions, we require that all *N* stocks that are randomly selected to construct portfolio have at least 120 months of overlapping return history. We construct portfolios by using a rolling-sample approach. We choose an estimation window of length *M* = 120 months. To avoid oversampling returns in the period of the US subprime crisis, we generate the first 120 months of portfolio returns.

Figure 4 (continued)

Panel A. Henriksson and Merton's (1981) timing coefficients

1) Fraction of times the portfolio shows a significant $\gamma_2 < 0$ at the 5% level (%)



2) Fraction of times the portfolio shows a significant $\gamma_1 > 0$ and $\gamma_2 < 0$ at the 5% level (%)

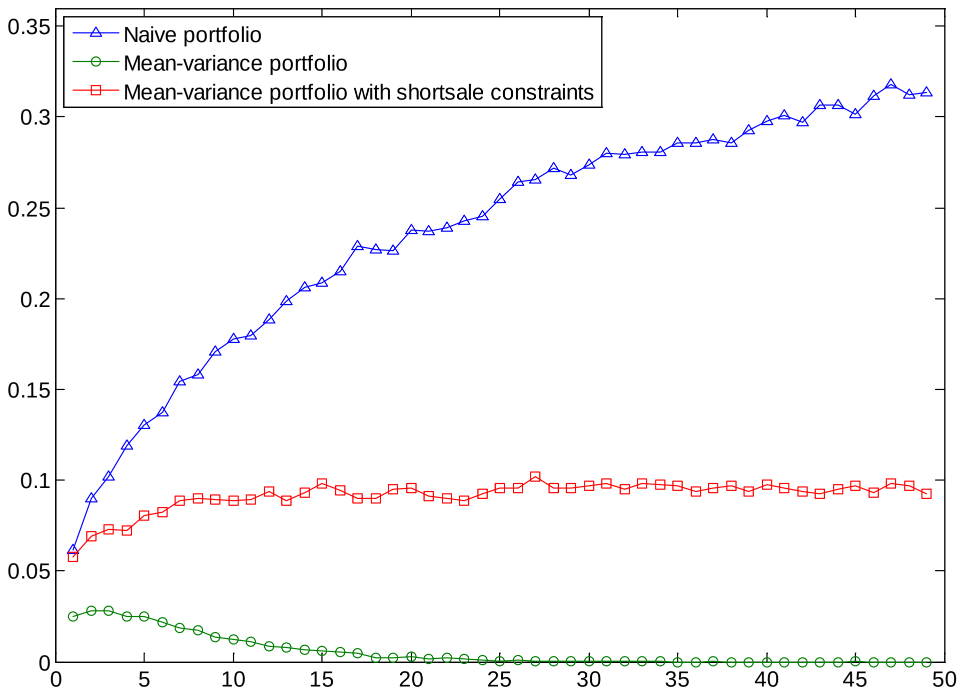
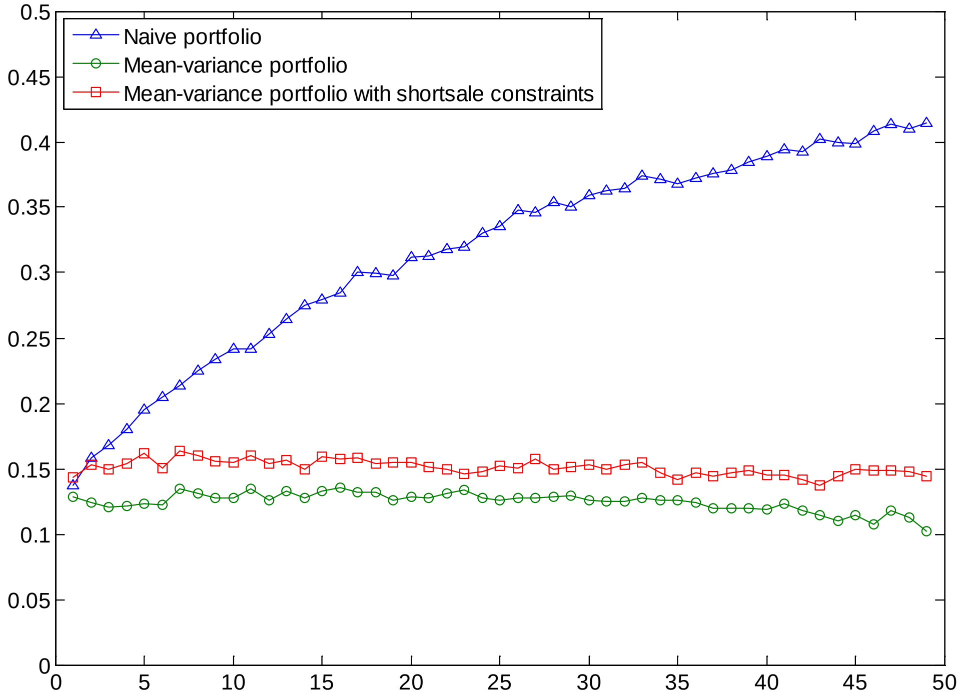


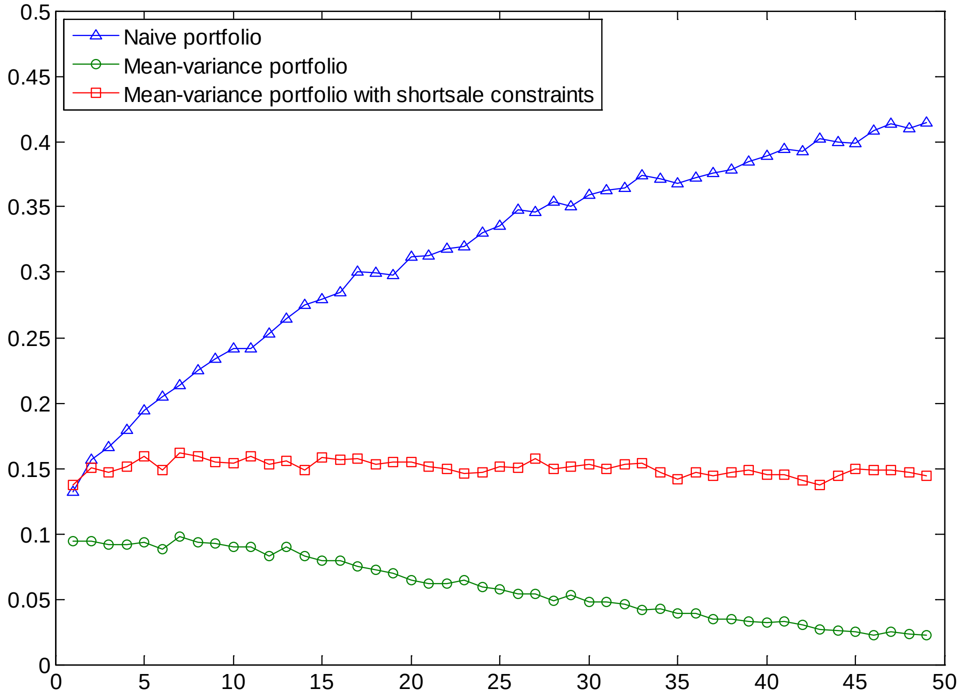
Figure 4 (continued)

Panel B. Treynor and Mazuy's (1966) timing coefficients

1) Fraction of times the portfolio shows a significant $\gamma_2 < 0$ at the 5% level (%)



2) Fraction of times the portfolio shows a significant $\gamma_1 > 0$ and $\gamma_2 < 0$ at the 5% level (%)


Naive portfolio
Mean-variance portfolio
Mean-variance portfolio with shortsale constraints
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
5
10
15
20
25
30
35
40
45
50